MATH CIRCLE 10/3/2009

PROBLEM SESSION IN PLANE GEOMETRY
Problems chosen from AMC,AIME, USAMO 2007

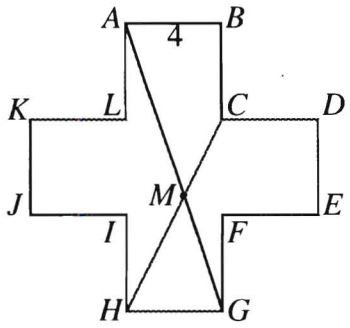
1) Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside ΔABC, $< ABC = 40^0$ and $< ADC = 140^0$ What is the degree measure of $< BAD$?

(A) 20 (B) 30 (C) 40 (D) 50 (E) 60

2) Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius 2. What is the area of the square?

(A) 32 (B) $22 + 12\sqrt{2}$ (C) $16 + 16\sqrt{3}$ (D) 48 (E) $36 + 16\sqrt{2}$.

3) Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that AG and CH meet at M. What is the area of quadrilateral $ABCM$?

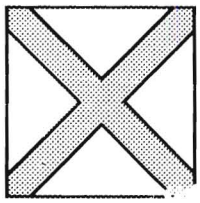


(A) 44/3 (B) 16 (C) 88/5 (D) 20 (E) 62/3

4) The point O is the center of the circle circumscribed about ΔABC, with $< BOC = 120^0$ and $< AOB = 140^0$. What is the degree measure of $< ABC$?

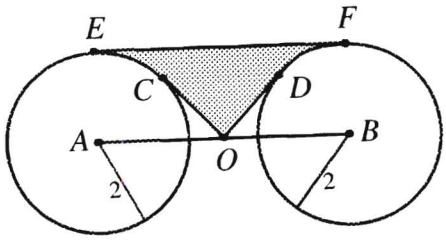
(A) 35 (B) 40 (C) 45 (D) 50 (E) 60

5) A paint brush is swept along both diagonals of a square to produce the symmetric painted area, as shown. Half the area of the square is painted. What is the ratio of the side length of the square to the brush width?



(A) $2\sqrt{2}+1$ (B) $3\sqrt{2}$ C) $2\sqrt{2}+2$ (D) $3\sqrt{2}+1$ (E) $3\sqrt{2}+2$

6) Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of AB, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and EF is a common tangent. What is the area of the shaded region $ECODF$?



A) $8\dfrac{\sqrt{2}}{3}$ (B) $8\sqrt{2}-4-\pi$ (C) $4\sqrt{2}$ D) $4\sqrt{2}+\dfrac{\pi}{8}$ (E) $8\sqrt{2}-2-\dfrac{\pi}{2}$

7) Two points B and C are in a plane. Let S be the set of all points A in the plane for which ΔABC has area 1. Which of the following describes S?

(A) two parallel lines (B) a parabola (C) a circle (D) a line segment
(E) two points

8) A circle passes through the three vertices of an isosceles triangle that has two sides of length 3 and a base of length 2. What is the area of this circle?

(A) 2π (B) $\dfrac{5\pi}{2}$ (C) $\dfrac{81}{32}\pi$ (D) 3π E) $\dfrac{7}{2}\pi$

9) A circle of radius 1 is surrounded by 4 circles of radius r. What is r ?

10) Right ΔABC has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in ΔABC with X and Y on AC, W on AB, and Z on BC. What is the side length of the square?

11) A triangle with side lengths in the ratio 3:4:5 is inscribed in a circle of radius 3. What is the area of the triangle?

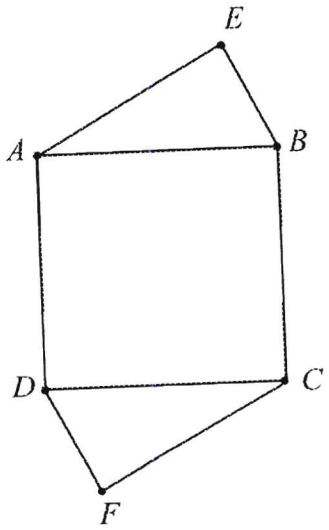
12) Triangles ABC and ADE have areas 2007 and 7002, respectively, with $B = (0; 0)$, $C = (223; 0)$, $D = (680; 380)$, and $E = (689; 389)$. What is the sum of all possible x-coordinates of A?

13) Point P is inside equilateral ΔABC. Points Q, R, and S are the feet of the perpendiculars from P to AB, BC, and CA, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB ?

14) Two particles move along the edges of equilateral ΔABC in the direction $A - B - C - A$; starting simultaneously and moving at the same speed. One starts at A, and the other starts at the midpoint of BC. The midpoint of the line segment joining the two particles traces out a path that encloses a region R. What is the ratio of the area of R to the area of ΔABC ?

15) How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

16) Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF.



17) In a right triangle ABC with right angle C, CA = 30 and CB = 16. Its legs CA and CB are extended beyond A and B. Points O_1 and O_2 lie in the exterior of the triangle and are the centers of two circles with equal radii. The circle with center O_1 is tangent to the hypotenuse and to the extension of leg CA, the circle with center O_2 is tangent to the hypotenuse and to the extension of leg CB, and the circles are externally tangent to each other. The length of the radius of either circle can be expressed as p/q, where p and q are relatively prime positive integers. Find p + q.